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COLORING SUBGRAPHS OF THE RADO GRAPH

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Given a universal binary countable homogeneous structure U and $n \in \omega$, there is a partition of the induced n-element substructures of U into finitely many classes so that for any partition $C_0, C_1, \ldots, C_{m-1}$ of such a class Q into finitely many parts there is a number $k \in m$ and a copy U^* of U in U so that all of the induced n-element substructures of U^* which are in Q are also in C_k.

The partition of the induced n-element substructures of U is explicitly given and a somewhat sharper result as the one stated above is proven.

1. Introduction

The Rado Graph $\mathbb{R} = (R; E)$ is the countable universal homogeneous graph. It is a countable graph with the defining property that for every finite set $F \subset R$ of vertices of the Rado graph and partition of F into the classes A and B there is a vertex x of the Rado graph which is adjacent to all vertices in A and not adjacent to any of the vertices in B. The injection $f: R \to R$ is an *embedding* of the Rado graph if x adjacent to y if and only if $f(x)$ adjacent to $f(y)$ for all vertices $x, y \in R$. The image of an embedding of R is a *copy* of $\mathbb{R}$.

It is just an exercise in the consequences of the defining property of the Rado graph, to show that the Rado graph is *indivisible*. That is, that for every partition A, B of the the vertices R of the Rado graph there exists a copy $\mathbb{R}^* = (R^*, E^*)$ of the Rado graph so that $R^* \subseteq A$ or $R^* \subseteq B$.

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How can we generalise this result to n-element subsets of $\mathbb{R}$? Let us consider the case $n = 2$. We could then ask the following question: If we colour the edges of the Rado graph with two colours, is there always an edge incidence preserving function of the edges to the edges so that all of the edges in the image of that function have the same colour? Note that such a function can map a triangle to a star. If the answer is yes, we could then say that the set of edges is indivisible. We do not consider this question. In this paper we stay within the category of embeddings of the ground set of the Rado graph to the ground set of the Rado graph.

Our aim is to extend the partition results of Erdős, Hajnal and Pósa, see [4], and of Pouzet and Sauer, see [7], dealing with two element subsets of the Rado graph to n-element subsets of the Rado graph and similar binary universal structures.

It is proven in [7] that there exists a partition of the two element subsets of the Rado graph into four equivalence classes. Two equivalence classes of edges and two equivalence classes of non-edges. If the two element subsets in any of the four classes is coloured with two colours, then there exists a copy of the Rado graph in which all of the coloured edges have the same colour. It is also shown there that this partition of the two element subsets of the Rado graph is best possible. It follows via standard arguments that if the two element subsets of the Rado graph are coloured with $n \in \omega$ colours then there exists a copy of the Rado graph in which only four of the colours appear, that is in which each of the classes of two element subsets is monochromatic.

Given a subset $Q \subseteq [R]^n$ of the n-element subsets of the base set R of the Rado graph. We say that Q is *indivisible* if for every colouring of the elements of Q with two colours there is a copy C of $\mathbb{R}$ in $\mathbb{R}$ so that $[C]^n \cap Q$ is coloured with only one of the two colours. This notion of indivisible is by now quite accepted. (The use of indivisible as defined might be misleading. It does not mean that for every colouring of the elements of Q with two colours there is a function of Q into Q, which in some sense preserves the structure of Q, and whose image is monochromatic.)

Naturally then the question arises to describe the different types of three element subsets of the Rado graph to obtain a partition into indivisible classes. We will provide an answer to this problem in the more general case of n element subsets. In fact we will define, for every $n \in \omega$, a partition of the n-element subsets of R into finitely many classes, so that each of the classes is indivisible. This result will provide the basis of a subsequent paper together with Laflamme and Vuksanovic in which we will prove that this partition is indeed best possible and in which we will use the present result to obtain a generalisation to infinitely many colours.

To define the different indivisible n-element types some properties of sets of finite sequences have to be defined. The results deal with the more general case of universal binary countable homogeneous structures. In the next section we will define those, make some general remarks on canonical partitions of relational structures, look at sets of finite sequences and state the result.

2. The result

Let $\mathfrak{L} = \{E_i \mid i \in n \in \omega\}$ be a finite list of binary relation symbols. The binary relational structure $\mathbb{A}$ with base set A is of *type* $\mathfrak{L}$ if each relation symbol $R \in \mathfrak{L}$ has an interpretation as a binary relation $R_{\mathbb{A}}$ on A. Let $\mathbb{A}$ and $\mathbb{B}$ be two relational structures of type $\mathfrak{L}$. The injection $f: A \to B$ is an *embedding* of $\mathbb{A}$ into $\mathbb{B}$ if $xR_{\mathbb{A}}y$ if and only if $f(x)R_{\mathbb{B}}f(y)$ for all $x, y \in A$ and all $R \in \mathfrak{L}$. An *isomorphism* of $\mathbb{A}$ to $\mathbb{B}$ is a bijective embedding. Those definitions of embedding and isomorphism extend in the obvious way to relational structures of arbitrary arity.

The $\mathfrak{L}$ structure $\mathbb{A}$ is an *induced substructure* of the $\mathfrak{L}$ structure $\mathbb{B}$ if the identity map of A is an embedding of $\mathbb{A}$ into $\mathbb{B}$. If $C \subseteq B$ then the *substructure of* $\mathbb{B}$ *induced by* C is the $\mathfrak{L}$ structure $\mathbb{C}$ with base set C which is an induced substructure of $\mathbb{B}$. The structure $\mathbb{A}^*$ is a *copy of* $\mathbb{A}$ *in* $\mathbb{B}$ if there exists an embedding f of $\mathbb{A}$ into $\mathbb{B}$ so that the substructure of $\mathbb{B}$ induced by $f[A]$ is isomorphic to $\mathbb{A}^*$. ($f[A] := \{f(a) \mid a \in A\}$.)

We will use the notation $\mathbb{A} = (A; \mathfrak{L})$ to indicate that $\mathbb{A}$ is a structure of type $\mathfrak{L}$ with base set A. Let $\mathbb{A} = (A; \mathfrak{L})$ and $x \in A$. Then $\mathbb{A} - x$ is the substructure of $\mathbb{A}$ induced by the set $A \setminus \{x\}$.

Let $\mathbb{A} = (A; \mathfrak{L})$ be a relational structure. Let Q be a set of finite subsets of A. The set Q is *indivisible within* $\mathbb{A}$ if for every partition $C_0, C_1, \dots, C_{m-1}$ of Q into $m \in \omega$ subsets there exists a copy $\mathbb{A}^* = (A^*; \mathfrak{L})$ of $\mathbb{A}$ in $\mathbb{A}$ so that all of the subsets of A^* which are in Q are in C_k.

The obvious first question then is whether the set of all n-element subsets, which induce a given substructure, is indivisible. It turns out that even for the set of one element subsets this is seldom the case. Hence the question becomes, given a relational structure $\mathbb{A}$, is there a partition of the n-element subsets of A into a finite number of classes each of which is indivisible.

The results of this paper apply not just to the Ramsey graph but to the more general countable universal binary structures. Let $\mathbf{F}$ be a set of relational structures of type $\mathfrak{L}$ whose base set is the set $\{0, 1\}$ and with the property that if $\mathbb{A}$ and $\mathbb{B}$ are two isomorphic relational structures in the language $\mathfrak{L}$ and set of elements $\{0,1\}$ then either both are in $\mathbf{F}$ or neither one of the two is in $\mathbf{F}$. Such a set $\mathbf{F}$ is a *universal constraint set.*

Let $\mathbf{F}$ be a universal constraint set. The relational structure $\mathbb{A}=(A,\mathfrak{L})$ *satisfies the constraint set* $\mathbf{F}$ if:

1. For any two elements $x,y\in A$ with $x\neq y$ the substructure of $\mathbb{A}$ induced by the set $\{x,y\}$ is isomorphic to one of the elments in $\mathbf{F}$.
2. $R(x,x)$ does not hold in $\mathbb{A}$ for any relation symbol $R\in\mathfrak{L}$ and any element $x\in A$. (This condition says that if $\mathbb{A}$ saitisfies the constraints then it does not have any "loops".)

The countable relational structure $\mathbb{U}_{\mathbf{F}}=(U,\mathfrak{L})$ is *universal of type* $\mathfrak{L}$ *under the constraints* $\mathbf{F}$ if it satisfies the constraints $\mathbf{F}$ and has the following *mapping extension property*:

Definition 2.1 (mapping extension property). For every finite relational structure $\mathbb{A}=(A;\mathfrak{L})$ which satisfies the constraints $\mathbf{F}$ and every element x in A and every embedding f of $\mathbb{A}-x$ into $\mathbb{U}_{\mathbf{F}}$ there is an extension of f to an embedding of $\mathbb{A}$ into $\mathbb{U}$.

Universal structures are special cases of homogeneous structures, see [10] for a more detailed description. Starting with a paper by Komjáth and Rödl [5], vertex partitions of homogeneous structures have been quite extensively studied, see [2], [3], [12]. Edge partitions of the Rado graph and the triangle free countable homogeneous graph are quite well understood, see [4], [7], [8]. The partition theory of the order structure of the rationals is completely solved, see [1] and [13].

In order to state the result we have to represent the elements of relational structures by sequences and have to define some notions for sets of finite sequences.

Let $n,m\in\omega$. We denote, for $m\in\omega+1$, by ${}^{n}m$ the set of all sequences $s=\langle s_0,s_1,\ldots,s_{n-1}\rangle$ of length n with entries s_i in m. Let $\mathfrak{T}_\omega:=\bigcup_{n\in\omega}{}^{n}\omega$. If $s=\langle s_0,s_1,\ldots,s_{n-1}\rangle\in\mathfrak{T}_\omega$ we denote by $|s|=n$ the *length* of s and write either s_i or $s(i)$ to denote the i's entry of the sequence s. The sequence t is an *initial segment* of the sequence s, written $t\subset s$, if $|t|<|s|$ and $t_i=s_i$ for every $i\in|t|$. We write $t\subseteq s$ if t is an initial segment of s or t is equal to s.

Given two sequences s and t we denote by $s\wedge t$, the *meet* of s and t, the longest sequence which is an initial segment of s and an initial segment of t. If $t\subseteq s$ then $s\wedge t=t$. The meet of two sequences always exists, it might be the empty sequence. Let $S\subseteq\mathfrak{T}_\omega$ be a set of sequences. The set closure(S) is the set S union the set of all meets of elements in S.

Let $s,t \in \mathfrak{T}_\omega$. Let $S \subseteq \mathfrak{T}_\omega$ and $t \in S$. The sequence s is an *immediate successor of t in S* if $t \subset s$ and there is no element $r \in S$ with $t \subset r \subset s$. The *degree of t in S* is the number of immediate successors of t in S.

The set S of sequences is an *antichain* if $x \subseteq y$ implies $x = y$ for all $x, y \in S$. The set S of sequences is *transversal* if $|x| = |y|$ implies $x = y$ for all $x, y \in S$.

Definition 2.2. The set $F \subseteq \mathfrak{T}_\omega$ of sequences is *diagonal* if it is an antichain and closure(F) is transversal and the degree of every element of closure(F) is at most two.

Definition 2.3. Let $s,t \in \mathfrak{T}_\omega$, then $x \prec y$ if x and y are imcomparable under $\subseteq$ and $x(|x \wedge y|) < y(|x \wedge y|)$. (Note that $\prec$ is not a total order.)

Definition 2.4. Let $R, S \subseteq \mathfrak{T}_\omega$ be two sets of sequences. The function f of R to S is a *similarity* of R to S if for all $x, y, z, u \in R$:

1. f is a bijection.
2. $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.
3. $|x \wedge y| < |z \wedge u|$ if and only if $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
4. If $|z| > |x|$ then $z(|x|) = f(z)(|f(x)|)$.
5. If $x \prec y$ then $f(x) \prec f(y)$.

The following might be helpful: Let R and S be two meet closed sets of sequences which are similar. Visualize both of them as planar tree drawings with the root at the bottom. Larger entries of a sequence make the corresponding branch turn further to the right. There are levels in the tree R which contain elements of R and levels which do not. The same for S. The trees R and S are "equal" after disregarding in R the levels which do not contain elements of R and in S levels which do not contain elements of S.

The sets R and S of sequences are *similar*, $R \sim S$, if there is a similarity of R to S. Note that if R is diagonal and R and S are similar then S is diagonal. We denote by $\mathrm{Sim}_R(S)$ the set of all subsets of R which are similar to S. The function f of R into $\mathfrak{T}_\omega$ is a *similarity embedding* if f is a similarity of R to $f[R]$.

Note that Item 1. of Definition 2.4 follows from Item 2. and that the composition of similarities is again a similarity and the inverse of a similarity is again a similarity. Hence $\sim$ is an equivalence relation on $\mathfrak{T}_\omega$.

In the next paragraphs we establish a correspondence between sets of sequences and universal binary structures.

Let Let $\mathbf{F}$ be a universal constraint set with $|\mathbf{F}| = k \in \omega$ and λ a bijection of $\mathbf{F}$ to k. We say that λ is a labeling of $\mathbf{F}$. Let $\mathbb{U}_{\mathbf{F}} = (U; \mathfrak{L})$ be a universal countable binary relational structure with constraints $\mathbf{F}$. Let $(u_i; i \in \omega)$ be an enumeration of the elements of U.

For every pair (u_n, u_m) with $n<m$ let $\mu(u_n, u_m) := \mathbb{F} \in \mathbf{F}$ be the structure in $\mathbf{F}$ for which the function mapping 0 to u_n and 1 to u_m is an isomorphism of $\mathbb{F}$ to the substructure of $\mathbb{U}_{\mathbf{F}}$ induced by $\{u_n, u_m\}$.

We associate with every element u_n of U a sequence σ_{u_n} of length n so that for every $i \in n$ the i's entry $\sigma_{u_n}(i) := \lambda(\mu(u_n, u_m))$. If F is a subset of U then $\sigma(F) := \{\sigma_x \mid x \in F\}$. If F and G are two subsets of U then $F \sim G$ if $\sigma(F) \sim \sigma(G)$ and F is *diagonal* if $\sigma(F)$ is diagonal. The sets F and G are *similar* if $F \sim G$. For $R \subseteq U$ we denote by $\mathrm{Sim}_R(F)$ the set of subsets G of R with $F \sim G$. Note that $\sim$ is an equivalence relation which partitions $[U]^n$ into finitely many similarity classes for every $n \in \omega$.

Let T be the subtree of $\mathfrak{T}_\omega$ consisting of all sequences with entries in $k = |\mathbf{F}|$. We define a relational structure of type $\mathfrak{L}$ on T to obtain the relational structure $\mathbb{T}_{\mathbf{F}} = (T, \mathfrak{L})$ as follows: Let t and s be two sequences in T with $|t| > |s|$. Let $\mathbb{F} \in \mathbf{F}$ be such that $\lambda(\mathbb{F}) = t(|s|)$. Then the function which maps 0 to s and 1 to t is an isomorphism of the structure $\mathbb{F}$ to the substructure of $\mathbb{T}_{\mathbf{F}}$ induced by the set $\{s, t\}$. It follows that σ is an isomorphic embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{T}_{\mathbf{F}}$.

Let $\mathbb{U}_{\mathbf{F}} = (U; \mathfrak{L})$ be a universal countable binary relational structure. The notions of similarity between subsets of U and of a subset of U being diagonal depend on the enumeration. We always assume that U is enumerated into an ω sequence and that the notions of similar and diagonal are relative to this fixed enumeration.

The injection f of $\mathbb{U}_{\mathbf{F}} = (U, \mathfrak{L})$ into $\mathbb{U}_{\mathbf{F}}$ is a *diagonalization* of $\mathbb{U}_{\mathbf{F}}$ if $f[U]$ is diagonal and $f[F] \sim F$ for every diagonal subset F of U. Let f be a diagonalization of $\mathbb{U}_{\mathbf{F}}$. The copy of $\mathbb{U}$ induced by $f[U]$ is a *diagonal representation of* $\mathbb{U}$.

Let f be a diagonalization of $\mathbb{U}_{\mathbf{F}}$ and $n < m \in \omega$. The set $\{u_n, u_m\}$ is a diagonal subset of U. It follows that $\mu(u_n, u_m) = \mu(f(u_n), f(u_m))$ and hence that f is an isomorphic embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{U}_{\mathbf{F}}$. $(\sigma_{u_m}(|\sigma_{u_n}|) = \sigma_{f(u_m)}(|\sigma_{f(u_n)}|).)$

Hence every diagonal representation of $\mathbb{U}_{\mathbf{F}}$ is isomorphic to $\mathbb{U}_{\mathbf{F}}$. That is, every diagonalization of $\mathbb{U}_{\mathbf{F}}$ is an embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{U}_{\mathbf{F}}$. It follows that if a diagonal representation of $\mathbb{U}_{\mathbf{F}}$ has a partition of it's n-element subsets into finitely many indivisible classes then $\mathbb{U}_{\mathbf{F}}$ has a partition of it's n-element subsets into finitely many indivisible classes.

We will prove in Section 7:

Theorem 2.1. *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and F a finite diagonal subset of U. Let $C_0, C_1, \ldots, C_{m-1}$ be a partition of $\mathrm{Sim}_U(F)$ into equivalence classes.*

Then there exists $k \in m$ *and a diagonalization* f *of* $\mathbb{U}$ *into* $\mathbb{U}$ *so that* $\mathrm{Sim}_{f[U]}(F) \subseteq C_k$.

Corollary 2.1. *Let* $\mathbb{U} = (U; \mathfrak{L})$ *be a universal countable binary relational structure and* F *a finite diagonal subset of* U. *The set* $\mathrm{Sim}_U(F)$ *is indivisible.*

Corollary 2.2. *Let* $\mathbb{V} = (V; \mathfrak{L})$ *be a diagonal representation of the universal countable binary relational structure* $\mathbb{U} = (U; \mathfrak{L})$ *and* F *a finite diagonal subset of* V. *Let* $C_0, C_1, \dots, C_{m-1}$ *be a partition of* $\mathrm{Sim}_V(F)$ *into equivalence classes.*

Then there exists $k \in m$ *and a diagonalization* f *of* $\mathbb{V}$ *into* $\mathbb{V}$ *so that* $\mathrm{Sim}_{f[V]}(F) \subseteq C_k$.

Proof. Let $|F| = n$ and let h be a diagonalization of $\mathbb{U}$ so that $h[U] = V$. For $i \in m$ let $C'_i = \{G \in \mathrm{Sim}_U(F) \mid h[G] \in C_i\}$. The sets C'_i form a partition of $\mathrm{Sim}_U(F)$. According to Theorem 2.1 there is a diagonalization g of $\mathbb{U}$ and $k \in m$ so that $\mathrm{Sim}_{g[U]}(F) \subseteq C'_k$.

The set $g[U]$ is diagonal and hence every subset of $g[U]$ is diagonal. It follows that $G \in \mathrm{Sim}_{h \circ g[U]}(F)$ if and only if there is $G' \in \mathrm{Sim}_{g[U]}(F)$ with $h[G'] = G$. The diagonalization h maps elements of C'_k to elements in C_k. Let $f = h \circ g$ restricted to V. ∎

Corollary 2.3. *Let* $\mathbb{V} = (V; \mathfrak{L})$ *be a diagonal representation of the universal countable binary relational structure* $\mathbb{U} = (U; \mathfrak{L})$ *and let* $n \in \omega$. *The* n *element subsets of* V *and hence the* n*-element subsets of* U *have a partition into finitely many indivisible equivalence classes.*

3. Prelimiaries

Let $s, t \in \mathfrak{T}_\omega$ and $t \subset x$. We will write s as $\langle t; s_{|t|}, s_{|t|+1}, \dots, s_{|s|-1}\rangle$. In particular $\langle t; l\rangle$ for $t \in \mathfrak{T}_\omega$ and $l \in \omega$ denotes the sequence s of length $|t|+1$, initial segment t and $s_{|t|} = s_{|s|-1} = l$. The set S is *closed under initial segments* if for every $s \in S$ and $i \in |s|$ the sequence $t \in S$ with $t \subset s$ and $|t| = i$ is also an element of S.

Let S and T be two meet closed subsets of $\mathfrak{T}_\omega$ and $f : S \to T$ a function of S to T. The function f is *meet preserving* if $f(s \wedge r) = f(s) \wedge f(r)$ for any two elements s and r of S. If f is meet preserving and an injection then f^{-1} is meet preserving and $x \subset y$ if and only if $f(x) \subset f(y)$.

The set $T \subseteq \mathfrak{T}_\omega$ of finite sequences is an ω*-tree* if it is nonempty, closed under initial segments, has no endpoints and every element of T has finite degree. Note that every ω-tree is closed under meet. The subset D of T is *cofinal in* T if for every $t \in T$ there is an element $d \in D$ with $t \subseteq d$.

The sequence l is *monotone* if $l_i < l_j$ for all $i,j \in |l|$ with $i<j$. Let $s,l \in \mathfrak{T}_\omega \cup {}^\omega\omega$, the sequence l monotone and let $m \in \omega$ be minimal so that $s(l(m))$ is undefined, that is so that $m \geq |l|$ or $|s| \geq l(m)$. The *composition* of the sequences s and l is the sequence $s\circ l$ of length m for which $(s\circ l)(i) = s(l(i))$ for all $i \in m$. (That is, composition of sequences is just function composition.)

It follows that the sequence $s \circ l$ is the subsequence of all of those entries s_i of s for which i is an entry of l. Which is of course the same as the subsequence obtained from s by removing all the entries with indices not in l.

Let S be a set of sequences. The sequence $\overrightarrow{\text{levels}}(S)$ is obtained by ordering the elements of the set $\text{levels}(S) := \{|s| \mid s \in S\}$ in strictly increasing order.

If S is a set of sequences and l monotone then $S \circ l := \{s \circ l \mid s \in S\}$. Note that the subsequence $s \circ \overrightarrow{\text{levels}}(S)$ is obtained from s by removing all of those entries s_i for which there is no element $t \in S$ with $|t| = i$. (If $i \in \omega$ and s a sequence with $i \geq |s|$ then the sequence obtained from s by removing the entry with index i is s.)

Let $S \subseteq \mathfrak{T}_\omega$. The function ν_S of S to $\mathfrak{T}\omega$ given by $\nu_S(x) = x \circ \overrightarrow{\text{levels}}(\text{closure}(S))$ is the *normal function* of S. For $i \in \omega$ let $\nu_{S,i}$ be the function which removes from every element $x \in S$ the entry with index i. It follows that ν_S can be expressed as a product $\nu_S = \nu_{S_{n-1},i_{n-1}} \circ \nu_{S_{n-2},i_{n-2}} \circ \cdots \circ \nu_{S_1,i_1} \circ \nu_{S_0,i_0}$ with $S_0 = S$ and i_j some number not in $\text{levels}(\text{closure}(S_{i_j}))$ and $S_{j+1} = \nu_{S_j,i_j}(S_j)$ for all $j \in n$.

Example 3.1. Let $S := \{\langle 3,1\rangle, \langle 3,1,0,5\rangle, \langle 3,1,2,4,6\rangle\}$.

Then S is closed and $\overrightarrow{\text{levels}}(\text{closure}(S)) = \overrightarrow{\text{levels}}(S) = \langle 2,4,5\rangle := l$ and hence $\nu_S[S] = \{\langle\,\rangle, \langle 0\rangle, \langle 2,6\rangle\} = S \circ l$. Then $\nu_{S,3} = \{\langle 3,1\rangle, \langle 3,1,0\rangle, \langle 3,1,2,6\rangle\}$. Let $R := v_{S,3}$ and $P := \nu_{R,1} = \{\langle 3\rangle, \langle 3,0\rangle, \langle 3,2,6\rangle\}$. Then we obtain finally $\nu_{P,0} = \{\langle\,\rangle, \langle 0\rangle, \langle 2,6\rangle\} = \nu_S[S]$.

Note that the function $\nu_S : S \to \mathfrak{T}_\omega$ is an injection. For assume that $s,t \in S$ with $s \neq t$. If $\{s,t\}$ is an antichain then $s(|s \wedge t|) \neq t(|s \wedge t)$ and both $s(|s \wedge t|)$ and $t(|s \wedge t|)$ are defined. Hence $\nu_S(s) \neq \nu_S(t)$. If $s \subset t$ then $s \wedge t = s$ and s does not have an entry with index $|s|$ while t does have an entry with index $|s|$. Hence $\nu_S(s) \neq \nu_S(t)$.

Example 3.2. Let $S = \{\langle 3,1,0,5\rangle, \langle 3,2,1,1\rangle\}$.

Then $\overrightarrow{\text{levels}}(S) = \langle 4\rangle := l$ and $\langle 3,1,0,5\rangle \circ l = \langle\ \rangle = \langle 3,2,1,1\rangle \circ l$.

Definition 3.1. Let $R,S \subseteq \mathfrak{T}_\omega$ be two sets of sequences. The function f of R to S is a *strong similarity* of R to S if for all $x,y,z,u \in R$:

1. f is a bijection.
2. $x \wedge y \subseteq z \wedge u$ if and only if $f(x) \wedge f(y) \subseteq f(z) \wedge f(u)$.
3. $|x \wedge y| < |z \wedge u|$ if and only if $|f(x) \wedge f(y)| < |f(z) \wedge f(u)|$.
4. If $|z| > |x \wedge y|$ then $z(|x \wedge y|) = f(z)(|f(x) \wedge f(y)|)$.

The sets R and S of sequences are *strongly similar* if there is a strong similarity of R to S. We write $R \overset{s}{\sim} S$ to indicate that R and S are strongly similar. If $S \subseteq T$ and f is a strong similarity of R to S then f is a *strong similarity embedding* of R into T and S is a *strong similarity copy* of R in T.

The notion of strongly similar sets of sequences is central to this paper. If F is a subset of the set R of sequences then $\mathrm{Sims}_R(F)$ is the set of all subsets of R which are strongly similar to F.

As an easy consequence of the definition of strong similarity we obtain the following Lemma:

Lemma 3.1. *Let $R, S, C, D \subseteq \mathfrak{T}_\omega$ be sets of sequences and f a strong similarity of R to S then:*

1. *f is a similarity of R to S.*
2. *If $x, y, z, u \in R$ then $x \wedge y = z \wedge u$ if and only if $f(x) \wedge f(y) = f(z) \wedge f(u)$.*
3. *The strong similarity f has a unique extension f' to a strong similarity of $\mathrm{closure}(R)$ to $\mathrm{closure}(S)$ via $f'(x \wedge y) = f(x) \wedge f(y)$ and f' is a meet preserving function of R.*
4. *If R and S are meet closed then f is a meet preserving bijection of R to S.*
5. *The product of strong similarities is a strong similarity and the inverse of a strong similarity is a strong similarity.*
6. *If $Q \subseteq R$ then the restriction of f to Q is a strong similarity of Q to $f[Q]$.*
7. *If $i \notin \mathrm{levels}(\mathrm{closure}(C))$ then $\nu_{C,i}$ is a strong similarity and $\nu_{C,i}[\mathrm{closure}(C)] = \mathrm{closure}(\nu_{C,i}[C])$.*
8. *The function ν_C is a strong similarity and $\nu_C[\mathrm{closure}(C)] = \mathrm{closure}(\nu_C[C])$.*
9. *If $C \subseteq \mathfrak{T}_\omega$ is meet closed and $l = \overrightarrow{\mathrm{levels}}(C)$ and $s, r \in C$ then $s = r$ if and only if $\nu_C(s) = \nu_C(r)$ if and only if $s \circ l = r \circ l$.*
10. *If $R = \nu_R[R]$ and $S = \nu_S[S]$ then $R = S$ and f is the identity map on R.*
11. *A bijection g of C to D is a strong similarity if and only if*
$$g = \nu_D^{-1} \circ \nu_C$$
if and only if
$$r \circ \overrightarrow{\mathrm{levels}}(\mathrm{closure}(C)) = g(r) \circ \overrightarrow{\mathrm{levels}}(\mathrm{closure}(D)) \text{ for all } r \in C.$$

12. The sets C and D of sequences are strongly similar if and only if

$$C \circ \overrightarrow{\text{levels}}(\text{closure}(C)) = D \circ \overrightarrow{\text{levels}}(\text{closure}(D)).$$

13. There is at most one strong similarity of R to S.
14. The set C of sequences is strongly similar to the set D of sequences if and only if $\nu_C[C]=\nu_D[D]$.

Definition 3.2. The set $F\subseteq\mathfrak{T}_\omega$ of sequences is *strongly diagonal* if it is an antichain and closure(F) is transversal and for all $x,y,z\in F$ with $x\neq y$:

1. $|x\wedge y|<|z|$ and $x\wedge y\not\subset z$ implies $z(|x\wedge y|)=0$.
2. $x(|x\wedge y|)\in\{0,1\}$.

It follows that every subset of a strongly diagonal set is strongly diagonal. Note that Item 2. of Definition 3.2 implies that the degree of every element of closure(F) is at most two and hence that every strongly diagonal set is diagonal.

Lemma 3.2. *Let F be a diagonal subset of $\mathfrak{T}_\omega$. If f is a function of F to $\mathfrak{T}_\omega$ so that for all $x,y,z,u\in F$*

$$|x\wedge y|<|z\wedge u| \text{ implies } |f(x)\wedge f(y)|<|f(z)\wedge f(u)| \tag{1}$$

then $f[F]$ is diagonal and for all $x,y,z,u\in F$:

1. $|f(x)\wedge f(y)|<|f(z)\wedge f(u)|$ implies $|x\wedge y|<|z\wedge u|$.
2. $x\wedge y\subseteq z\wedge u$ if and only if $f(x)\wedge f(y)\subseteq f(z)\wedge f(u)$.

Proof. That f is an injection and closure($f[F]$) transversal and that the degree of every element in closure($f[F]$) is at most two follows easily. The set $f[F]$ is an antichain because if $x\neq y$ are elements of the antichain F then $|x\wedge y|<|x|$ and $|x\wedge y|<|y|$ and hence we obtain from condition (1) that $|f(x)\wedge f(y)|<|f(x)|$ and $|f(x)\wedge f(y)|<|f(y)|$.

Let $|f(x)\wedge f(y)|<|f(z)\wedge f(u)|$ and assume for a contradiction that $|x\wedge y|=|z\wedge u|$. Then $x\neq y$ and $z\neq u$ because closure(F) transversal. If $x=z$ then $|x\wedge y|=|x\wedge u|$ and $y\neq u$ and $|x\wedge y|<|u\wedge y|$ because the degree of $x\wedge y$ in F is at most two. It follows that $|f(x)\wedge f(y)|<|f(u)\wedge f(y)|$ and hence $|f(x)\wedge f(y)|=|f(x)\wedge f(u)|$. If the x,y,z,u are pairwise different we may assume without loss that $|x\wedge y|<|x\wedge z|$ and $|x\wedge y|<|u\wedge y|$, which using (1) implies $|f(x)\wedge f(y)|<|f(x)\wedge f(z)|$ and $|f(x)\wedge f(y)|<|f(u)\wedge f(y)|$, which in turn implies that $|f(x)\wedge f(y)|=|f(z)\wedge f(u)|$.

Let $x\wedge y\subseteq z$. If $x=y=z$ then $f(x)\wedge f(y)\subseteq f(z)$ follows. Otherwise $x\neq y$ and the degree of $x\wedge y$ in closure(F) is two and $x\wedge y\subset z$ because F is an antichain and the degree of $x\wedge y$ is at most two. The fact that the degree of $x\wedge y$ is two and $x\wedge y\subset z$ imply that $x\wedge y\subset x\wedge z$ or $x\wedge y\subset y\wedge z$. Say, $x\wedge y\subset x\wedge z$, which implies due to condition (1) that $|f(x)\wedge f(y)|<|f(x)\wedge f(z)|$ which in turn implies $f(x)\wedge f(y)\subset f(x)\wedge f(z)\subseteq f(z)$.

If $x\wedge y\subseteq z\wedge u$ then $x\wedge y\subseteq z$ and $x\wedge y\subseteq u$ and hence $f(x)\wedge f(y)\subseteq f(z)$ and $f(x)\wedge f(y)\subseteq f(u)$ and hence $f(x)\wedge f(y)\subseteq f(z)\wedge f(u)$.

If $f(x)\wedge f(y)\subseteq f(z)$ then $x\wedge y\subseteq z$ because the set $f[F]$ of sequences and the function f^{-1} satisfy the conditions of the Lemma. ∎

Lemma 3.3. *Let $F\subseteq\mathfrak{T}_\omega$ be a strongly diagonal set and f a strong similarity embedding of F into $\mathfrak{T}_\omega$. Then $f[F]$ is a strongly diagonal set.*

Proof. The strongly diagonal set F of sequences and the strong similarity f satisfy the conditions of Lemma 3.2. Hence $f[F]$ is diagonal. Item 1. of Definition 3.2 follows from Item 4. of Definition 3.1 together with Item 2. of Lemma 3.2. Item 2. of Definition 3.2 follows from Item 4. of Definition 3.1. ∎

Lemma 3.4. *Let $F\subseteq\mathfrak{T}_\omega$ be a diagonal set and f a function of F into $\mathfrak{T}_\omega$. The function f is a strong similarity embedding if and only if for all $x,y,z,u\in F$:*

1. *$|x\wedge y|<|z\wedge u|$ implies $|f(x)\wedge f(y)|<|f(z)\wedge f(u)|$.*
2. *If $|z|>|x\wedge y|$ then $z(|x\wedge y|)=f(z)(|f(x)\wedge f(y)|)$.*

Proof. Item 1. implies that f is a bijection and Items 2. and 3. of Definition 3.1 follow from Lemma 3.2 and Item 4. is the present Item 2. ∎

Definition 3.3. Let S and T be two subsets of $\mathfrak{T}_\omega$. The injection f of S to T is a *strong diagonalization* of S to T if for all $x,y,z,u\in S$:

1. The set of sequences $f[S]$ is strongly diagonal.
2. $|x\wedge y|<|z\wedge u|$ implies $|f(x)\wedge f(y)|<|f(z)\wedge f(u)|$.
3. If $|x|>|y|$ then $x(|y|)=f(x)(|f(y)|)$.
4. If $x\prec y$ then $f(x)\prec f(y)$.

Lemma 3.5. *Let f be a similarity of the diagonal set F to $\mathcal{T}_\omega$ and $x,y,z,u\in F$. Then*

$$x\wedge y\subseteq z\wedge u \text{ if and only if } f(x)\wedge f(y)\subseteq f(z)\wedge f(u).$$

Proof. Follows readily from Lemma 3.2. ∎

Definition 3.4. Let $S,T\subseteq\mathfrak{T}_\omega$. The function $f: S\to T$ is a *strong $\mathfrak{d}$-morphism* if for every strongly diagonal subset F of S the restriction of f to F is a strong similarity embedding of F into T.

Definition 3.5. Let $S,T\subseteq\mathfrak{T}_\omega$. The function $f:S\to T$ is a $\mathfrak{d}$-*morphism* if for every diagonal subset F of S the restriction of f to F is a similarity embedding of F into T.

Lemma 3.6. *If f is a similarity of the strongly diagonal set F to the strongly diagonal set G then f is a strong similarity.*

Proof. Let f be a similarity of S to T and F a strongly diagonal subset of S. We have to verify Item 4. of Definition 3.1. Let $|z|>|x\wedge y|$. If $x=y$ then Item 4. of Definition 3.1 follows from Item 4. of Definition 2.4.

If $x\neq y$ and $x\wedge y\not\subset z$ then $z(|x\wedge y|)=0$ because F is strongly diagonal. It follows from Lemma 3.5 that $f(x)\wedge f(y)\not\subset f(z)$ and hence that $f(z)(|f(x)\wedge f(y)||)=0$ because $f[F]=G$ is strongly diagonal. ($f(x)\wedge f(y)\neq f(z)$ otherwise $f[F]$ would not be an antichain.)

If $x\neq y$ and $x\wedge y\subset z$. Then $\{x(|x\wedge y|),y(|x\wedge y|)\}=\{0,1\}$ and $y\not\subset x$ and $x\not\subset y$ because F is an antichain. We may assume without loss that $x(|x\wedge y|)=0$ and $y(|x\wedge y|)=1$. Hence $x\prec y$ and we obtain from Definition 2.4 that $f(x)\prec f(y)$. Because G is strongly diagonal it follows that $\{f(x)(|f(x)\wedge f(y)|),f(y)(|f(x)\wedge f(y)|)\}=\{0,1\}$ which together with $f(x)\prec f(y)$ implies $f(x)(|f(x)\wedge f(y)|)=0$ and $f(y)(|f(x)\wedge f(y)|)=1$. Finally $x(|x\wedge y|)=f(x)(|f(x)\wedge f(y)|)$ and $y(|x\wedge y|)=f(y)(|f(x)\wedge f(y)|)$.

It follows from $|z|>|x\wedge y|$ and the fact that the degree of $|x\wedge y|$ in F is two that $z(|x\wedge y|)=x(|x\wedge y|)$ or $z(|x\wedge y|=y(|x\wedge y|)$. Say $z(|x\wedge y|)=x(|x\wedge y|)$. Then it follows from the fact that the degree of $x\wedge y$ is two that $x\wedge y\subset z\wedge x$. Using Lemma 3.5 and Item 3. of Definition 2.4 we obtain $f(x)\wedge f(y)\subset f(z)\wedge f(x)$ which in turn implies that $f(x)(|f(x)\wedge f(y)|)=f(z)(|f(x)\wedge f(y)|)$.

Hence $z(|x\wedge y|)=x(|x\wedge y|)=f(x)(|f(x)\wedge f(y)|)=f(z)(|f(x)\wedge f(y)|)$. ∎

Lemma 3.7. *Every strong diagonalization is a $\mathfrak{d}$-morphism.*

Proof. Let f be a strong diagonalization of S to T and F a diagonal subset of S. Item 2. of Definition 2.4 follows from Item 2. of Lemma 3.2 and Item 3. of Definition 2.4 from Item 1. of Lemma 3.2.

Item 4. of Definition 2.4 follows from Item 3. of Definition 3.3. Item 5. of Definition 2.4 follows from Item 4. of Definition 3.3. ∎

Corollary 3.1. *If $F\subseteq\mathfrak{T}_\omega$ is diagonal and f a strong diagonalization of F into $\mathfrak{T}_\omega$ then $f[F]$ is strongly diagonal and $F\sim f[F]$.*

Corollary 3.2. *Every strong diagonalization is a strong $\mathfrak{d}$-morphism.*

Proof. Follows from Lemma 3.6 and Lemma 3.7. ∎

The following Lemma is easily checked.

Lemma 3.8. *If f is a strong similarity of R to S and g a strong diagonalization of S to T then $g\circ f$ is a strong diagonalization of R to T. If f is a strong diagonalization of R to S and g a strong similarity of S to T then $g\circ f$ is a strong diagonalization of R to T.*

4. The strong diagonalization theorem

Definition 4.1. The ω-tree T is *wide*, if:

a. $t\in T$ implies $\langle t;0\rangle\in T$ and $\langle t;1\rangle\in T$.
b. $i\in n\in\omega$ and $\langle t;n\rangle\in T$ implies $\langle t;i\rangle\in T$.
c. If $s,t\in T$ and $|s|<|t|$ then the degree of $s\in T$ is less than or equal to the degree of t in T.

There are three ways of understanding the proof of the following Theorem 4.1. The first is to study the definition of strong diagonalization carefully and then to see that there is certainly enough room in a wide omega tree T to accommodate a strong diagonalization of T into T. The second one is to read the proof of Theorem 4.1 to the end of the construction of the function f and then to see that there is certainly enough room in a wide omega tree T to proceed with an induction argument. The third one is to read through the gory details.

Theorem 4.1. *Let T be a wide ω-tree and D a cofinal subset of T. Then there exists a strong diagonalization f of T into D.*

Proof. For $n\in\omega$ let $T(n)\subseteq T$ be the set of sequences of length n and let $T(\leq n)$ be the set of sequences in T of length at most n. Note that the root of T is the empty sequence $\langle\rangle$, that $T(0)=\{\langle\rangle\}$ and that $T(1)$ is the set of sequences in T which contain exactly one element.

We will use the notation $f[T(n)]:=\{f(s)\mid s\in T(n)\}$ and for $S,R\subseteq T$ and $n\in\omega$ the expression $S<R$ to mean $|s|<|r|$ for all $s\in S$ and $r\in R$ and $S<n<R$ to mean that $|s|<n<|r|$ for all $s\in S$ and $r\in R$. If s is an immediate successor of t then t is the *immediate predecessor* of s. We denote the immediate predecessor of a sequence s by $\dot{s}$.

We will construct the function f so that it is a strong diagonalization of T into D and a function g of $T\setminus\{\langle\rangle\}$ into T and a number $l(n)$ for every $n\in\omega$ which satisfy the following conditions:

i. $|s|<|f(s)|$ for all $s\in T$.
ii. $g[T(1)]<l(0)<f(\langle\rangle)<g[T(2)]<l(1)<f[T(1)]<g[T(3)]<l(2)$ and

$$g[T(n+1)]<l(n)<f[T(n)]<g[T(n+2)]<l(n+1)<f[T(n+1)]<$$
$$<g[T(n+3)]<l(n+2)<f[T(n+2)].$$

iii. If $s\in T$ and $|s|\geq 1$ then $g(s)$ is a predecessor of $f(\dot{s})$ with $g(s)=f(\dot{s})\wedge f(s)$ and $f(\dot{s})(|g(s)|)=0$ and $f(s)(|g(s)|)=1$. (To satisfy this condition we use Definition 4.1 of wide ω-tree.)
iv. If $|s|=|t|$ and $s\prec t$ then $|f(s)|<|f(t)|$ and $|g(s)|>|g(t)|$ and $f(s)\wedge f(t)=g(s)\wedge g(t)$.
v. For all $n\in\omega$. The difference between $l(n)$ and the largest element in $\{|f(t)|\,|\,t\in T(n)\}$ is larger than the number of elements in $T(n+2)$.
vi. Let $s\in T(n+1)$ and $n\in\omega$. Then $|f(s)|>l(n)$ and the entry $f(s)(i)$ of the sequence $f(s)$ is 0 for all indices i with $|g(s)|<i\leq l(n)$, unless there exists $t\in T(n)$ with $|f(t)|=i$ in which case $f(s)(i)=f(s)(|f(t)|)=s(|t|)$. (To satisfy this condition we use Definition 4.1 of wide ω-tree.) The first $l(0)$ entries of of the sequence $f(\langle\rangle)$ are 0.

Let $l(0)=|T(1)|+1$ and $f(\langle\rangle)$ be a sequence in D with $|f(\langle\rangle)|>l(0)$ and so that it's first $l(0)$ entries are equal to 0. Let $g[T(1)]$ be the set of predecessors of $f(\langle\rangle)$ so that $g[T(1)]<l(0)$ and if $s,t\in T(1)$ with $s\prec t$ then $|g(s)|>|g(t)|$. Note that the functions f and g defined so far satisfy Items i. to vi.

If the function f is given on $T(\leq n)$ and the function g is given on $T(\leq n+1)$ and the function l is given on all numbers $\leq n$ and if they and satisfy Items i. to vi. we extend the functions f and g and l as follows.

Let $l(n+1)$ be a number which satisfies Item v.

Then we determine for every $s\in T(n+1)$ a sequence $f(s)\in D$ so that all of those choices satisfy Items i., ii., iii., iv. and vi. That is, we choose the sequence $f(s)$ to be a successor of $g(s)$ so that $f(|g(s)|)=1$, then the next entries so that $f(s)$ satisfies Item vi. and long enough to satisfy Item ii. We make sure that if $s\prec t$ then $|f(s)|<|f(t)|$. Then it follows that $f(\dot{s})\wedge f(s)=g(s)$ because $g(s)$ is a predecessor of $f(\dot{s})$ with $f(\dot{s})(|g(s)|)=0$ and we have chosen $f(s)$ to be a successor of $g(s)$ with $f(s)(|g(s)|=1$. That $f(s)\wedge f(t)=g(s)\wedge g(t)$ we see as follows:

Item iv. implies that if $\dot{s}\neq\dot{t}$ then $f(\dot{s})\wedge f(\dot{t})=g(\dot{s})\wedge g(\dot{t})$ and hence we get from Item ii. that $|f(\dot{s})\wedge f(\dot{t})|<l(n-1)$. Hence $g(s)\not\subseteq g(t)\not\subseteq g(s)$. If $\dot{s}=\dot{t}$ let $s\prec t$ and hence $|g(s)|>|g(t)|$. Then $g(s)(|g(t)|)=f(\dot{s})(|g(t)|)=0$ and $f(s)(|g(t)|)=0$ and $f(t)(|g(t)|)=1$. Hence $f(s)\wedge f(t)=g(t)=g(s)\wedge g(t)$.

Finally we determine for every $s\in T(n+2)$ a predecessor $g(s)$ of $f(\dot{s})$ so that those choices satisfy Items ii. and iv. Note that $f(\dot{s})(|g(s)|)=0$ because $g(s)$ is a predecessor of $f(\dot{s})$ and we have chosen $f(\dot{s})$ so that all entries with with index in between $g(\dot{s})$ and $l(n+1)$ are 0. Hence Item iii. follows.

We procede with this construction and obtain functions f and g of T to D and a function l of ω to ω which satisfy Items i. to vi. It follows that closure$(f[T]) = f[T] \cup g[T]$ and $f[T] \cap g[T] = \emptyset$ and from Items ii. and iv. that f is an injection.

Claim 1. *The functions f and g satisfy:*

$$f(s) \wedge f(t) = \begin{cases} g(t) & \text{if } |s| = |t| \geq 1 \text{ and } \dot{s} = \dot{t} \text{ and } s \prec t\ , \\ f(\dot{s}) \wedge f(\dot{t}) & \text{if } |s| = |t| \geq 1 \text{ and } \dot{s} \neq \dot{t}, \\ f(\dot{s}) \wedge f(t) & \text{if } |s| \geq 1 \text{ and } \dot{s} \neq t \in T(\leq |s|-1), \\ g(s) & \text{if } |s| \geq 1 \text{ and } t = \dot{s}. \end{cases}$$

Proof of Claim 1. Let $|s| = |t| \geq 1$ and $\dot{s} = \dot{t}$ and $s \prec t$. Then, from Item iv., $f(s) \wedge f(t) = g(s) \wedge g(t)$ and $|g(s)| > |g(t)|$. Because, Item iii., both $g(s)$ and $g(t)$ are predecessors of $f(\dot{s})$, they are ordered under $\subset$ and hence $g(s) \wedge g(t) = g(t)$.

Let $|s| = |t| \geq 1$ and $\dot{s} \neq \dot{t}$. Then $f(s) \wedge f(t) = g(s) \wedge g(t)$. Because $g(s)$ is a predecessor of $f(\dot{s})$ and $g(t)$ is a predecessor of $f(\dot{t})$ and $g(s) \neq g(t)$ it follows that $g(s) \wedge g(t) = f(\dot{s}) \wedge f(\dot{t})$.

Let $|s| \geq 1$ and $\dot{s} \neq t \in T(\leq |s|-1)$. If $|t| = |s| - 1 = \dot{s}$ then $|f(\dot{s}) \wedge f(t)| = |g(\dot{s}) \wedge g(t)| < |g(s)|$, from Item ii. This together with $f(\dot{s}) \wedge f(s) = g(s)$ implies that $f(s) \wedge f(t) = f(\dot{s}) \wedge f(t)$. If $|t| < |s| - 1$ then we obtain from Item ii. that $|f(\dot{s}) \wedge f(t)| \leq |f(t)| < |g(s)|$.

The last case follows directly from Item iii. ∎

Claim 2. *If $s, t \in T$ with $s \neq t$ then $f(s) \wedge f(t) \in g[T]$. Also* closure$(f[T]) =$ *$f[T] \cup g[T]$ and $f[T] \cap g[T] = \emptyset$.*

Proof of Claim 2. The claim $f(s) \wedge f(t) \in g[T]$ follows readily from Claim 1 by induction on $\max(|s|, |t|)$. Item ii. implies $f[T] \cap g[T] = \emptyset$. ∎

Claim 3. *Let $n \in \omega$. If f restricted to $T(\leq n)$ is a strong diagonalization of $T(\leq n)$ into T then f restricted to $T(\leq n+1)$ is a strong diagonalization of $T(\leq n+1)$ into T.*

Proof of Claim 3. Assume that f restricted to $T(\leq n)$ is a strong diagonalization of $T(\leq n)$ into T.

The set $f[T(\leq n+1)]$ of sequences is strongly diagonal:

It follows from Items ii. and iv. that closure$(f[T]) = f[T] \cup g[T]$ is transversal.

If there are sequences $s, t \in T(\leq n+1)$ with $s \neq t$ and $f(t) \subset f(s)$ then $f(t) = f(s) \wedge f(t) \in g[T]$ according to and in contradiction to Claim 2. Hence $f[T]$ is an antichain.

Let $x, y, s \in T(\leq n+1)$ with $x \neq y$ and $|f(x) \wedge f(y)| < |f(s)|$ and $f(x) \wedge f(y) \not\subset f(s)$.

Let $s\in T(n+1)$. If $|f(x)\wedge f(y)|<|g(s)|$ then $f(x)\wedge f(y)\not\subset f(\dot{s})$ and hence $f(s)(|f(x)\wedge f(y)|)=f(\dot{s})(|f(x)\wedge f(y)|)=0$. If $|f(x)\wedge f(y)|\geq|g(s)|$ then $f(x)\wedge f(y)=g(t)$ for some $t\in T(n+1)$ with $s\neq t$ because $f(x)\wedge f(y)\not\subset f(s)$. We have $f(s)(|g(t)|)=0$ from Item vi. and the fact that closure($f[T]$) is transversal.

Let $s\in T(\leq n)$ and not both $x,y\in T(n+1)$ with $\dot{x}=\dot{y}$. Because $|f(x)\wedge f(y)|<|f(s)|$ it can not be the case that one of x and y, say y, is an element of $T(n+1)$ and $x=\dot{y}$. Hence it follows from Claim 1, that there are sequences $x',y'\in T(\leq n)$ so that $f(x)\wedge f(y)=f(x')\wedge f(y')$. Hence $f(s)(|f(x)\wedge f(y)|)=f(s)(|f(x')\wedge f(y')|)=0$ because $f[T(\leq n)]$ is strongly diagonal.

Let $s\in T(\leq n)$ and $|x|=|y|\in T(n+1)$ and $\dot{x}=\dot{y}$ and $x\prec y$. Then $f(x)\wedge f(y)=g(y)$ according to Claim 1. It follows from Items ii. and vi. that $f(s)(|g(y)|)=0$.

Finally, in order to show that $f[T]$ is strongly diagonal we have to prove that $f(x)(|f(x)\wedge f(y)|)\in\{0,1\}$ if $f(x)\neq f(y)$. If one of x and y, say y, is an element of $T(n+1)$ and $x=\dot{y}$ then $f(x)\wedge f(y)=g(y)$ and $f(x)(|g(y)|)=0$ and $f(y)(|g(y)|)=1$ according to Item iii. Let $x,y\in T(n+1)$ with $\dot{x}=\dot{y}$. If $x\prec y$ then $f(x)\wedge f(y)=g(y)$ and $f(x)(|g(y)|)=0$ from Item vi. and the fact that closure($f[T]$) is transversal. Accordint to Item iii. we have $f(y)(|g(y)|)=1$.

In all other cases it follows from Claim 1, that there are sequences $x',y'\in T(\leq n)$ with $f(x)\wedge f(y)=f(x')\wedge f(y')$ and $f(x')(|f(x')\wedge f(y')|)=f(x)(|f(x)\wedge f(y)|)$ and $f(y')(|f(x')\wedge f(y')|)=f(y)(|f(x)\wedge f(y)|)$.

For $s\in T(\leq n+1)$ let $\overline{s}:=s$ if $s\in T(\leq n)$ and let $\overline{s}=\dot{s}$ if $s\in T(n+1)$. It follows that if $|x\wedge y|<n$ then $f(x)\wedge f(y)=f(\overline{x})\wedge f(\overline{y})$. For if x and y are sequences in $T(\leq n)$ then $f(\overline{x})=f(x)$ and $f(\overline{y})=f(y)$. If $x\in T(n+1)$ and $y\in T(\leq n)$ then $f(\overline{x})\wedge f(\overline{y})=f(\dot{x})\wedge f(y)=f(x)\wedge f(y)$ according to Claim 1 unless $y=\dot{x}$ in which case $|x\wedge y|=|\dot{x}|=n$. If $x,y\in T(n+1)$ then $f(\overline{x})\wedge f(\overline{y})=f(\dot{x})\wedge f(\dot{y})=f(x)\wedge f(y)$ according to Claim 1, unless $\dot{x}=\dot{y}$ in which case $x\wedge y=x$ if $x=y$ and if $x\neq y$ then $x\wedge y=\dot{x}$. Hence in any case $|x\wedge y|\geq n$. Note that if $|x\wedge y|<n$ then $x\wedge y=\overline{x}\wedge\overline{y}$.

In order to establish Item 2. of the definition of strong diagonalization let $|x\wedge y|<|z\wedge u|$.

If $|x\wedge y|<n$ and $|z\wedge u|<n$ then $|\overline{x}\wedge\overline{y}|<|\overline{z}\wedge\overline{u}|$ and hence $|f(x)\wedge f(y)|=|f(\overline{x})\wedge f(\overline{y})|<|f(\overline{z}\wedge f(\overline{u}|=|f(z)\wedge f(u)|$.

If $z=u\in T(n)$ and $|x\wedge y|<n$ then $|\overline{x}\wedge\overline{y}|=|x\wedge y|<|z|$. Hence $|f(x)\wedge f(y)|=|f(\overline{x})\wedge f(\overline{y})|<|f(z)\wedge f(z)|$. If $z=u\in T(n+1)$ and $|x\wedge y|<n$ and $x\neq y$ then because $f(x)\wedge f(y)\in g[T]$ it follows from Item ii. that $|f(x)\wedge f(y)|<|f(z)|$. If $z=u\in T(n+1)$ and $|x\wedge y|<n$ and $x=y$ then $|f(x)|<|f(y)|$ follows from Item ii.

If $|z\wedge u|\geq n$ and $z\neq u$ then either $z\in T(n+1)$ and $u=\dot{z}$ or $u\in T(n+1)$ and $z=\dot{u}$ or $z,u\in T(n+1)$ and $\dot{z}=\dot{u}$. In either case there is $s\in\{z,u\}$ with $s\in T(n+1)$ and $|g(s)|\leq|f(z)\wedge f(u)|$. If $x=y$ with $|x|<n$ then $|f(x)|<|g(s)|$ according to Item ii. If $|x\wedge y|<n$ and $x\neq y$ then $|f(x)\wedge f(y)|=|f(\overline{x})\wedge f(\overline{y})|$. It follows from $|x\wedge y|<n$ that $n\geq 1$ and because $f(\overline{x})\wedge f(\overline{y})\in g[T]$ we obtain from Item ii. that $|f(\overline{x})\wedge f(\overline{y})|<l(n-1)$. Item ii. implies that $|g(s)|>l(n-1)$ because $s\in T(n+1)$.

If $|x\wedge y|\geq n$ and $|z\wedge u|\geq n$ then $|x\wedge y|=n$ and $|z\wedge u|=n+1$. Hence $z=u\in T(n+1)$. If $x=y\in T(n)$ then $|f(x)|<|f(z)|$ from Item ii. If $x\neq y$ then $x=\dot{y}$ or $\dot{x}=y$ or $\dot{x}=\dot{y}$. If $x=\dot{y}$ then $f(x)\wedge f(y)=g(y)$ according to Item iii. and we get $|g(y)|<|f(z)|$ from Item ii. If $\dot{x}=\dot{y}$ and $x\prec y$ then $f(x)\wedge f(y)=g(y)$ according to Claim 1.

Let $|x|>|y|$, then $|f(x)|>|f(y)|$. We wish to prove that $x(|y|)=f(x)(|f(y)|)$. In the case that $x\in T[\leq n]$ equality $x(|y|)=f(x)(|f(y)|)$ follows from the fact that f restricted to $T(\leq n)$ is a strong diagonalization. Let $x\in T(n+1)$ and $y\in T(n)$. Then $x(|y|)=f(x)(|f(y)|)$ follows from Item vi. Let $x\in T(n+1)$ and $y\in T(<n)$. Then $x(|y|)=\dot{x}(|y|)$ and $f(\dot{x})\wedge f(x)=g(x)$ from Item iii. with $|g(x)|>|f(y)|$ from Item ii. Hence $f(x)(|f(y)|)=f(\dot{x})(|f(y)|)$. We get $x(|y|)=\dot{x}(|y|)=f(\dot{x})(|f(y)|)=f(x)(|f(y)|)$.

Let $x\prec y$, then x and y are incomparable under $\subseteq$. If $x,y\in T(\leq n)$ then $f(x)\prec f(y)$ follows from the fact that f restricted to $T(\leq n)$ is a strong diagonalization.

If $x\in T(n+1)$ and $y\in T(\leq n)$ then we get from Claim 1 and the fact that $\dot{x}\neq y$ that $f(x)\wedge f(y)=f(\dot{x})\wedge f(y)$. This implies, together with Item ii. that $|g(x)|>|f(x)\wedge f(y)|$. Hence $x\prec y$ implies $\dot{x}\prec y$ implies $f(\dot{x})\prec f(y)$ implies $g(x)\prec f(y)$ implies $f(x)\prec f(y)$. The case $x\in T(\leq n)$ and $y\in T(n+1)$ is dual.

If $x,y\in T(n+1)$ and $\dot{x}\neq\dot{y}$ then $x\prec y$ implies $\dot{x}\prec\dot{y}$ implies $f(\dot{x})\prec f(\dot{y})$ implies $g(x)\prec g(y)$ implies $f(x)\prec f(y)$. That $f(\dot{x})\prec f(\dot{y})$ implies $g(x)\prec g(y)$ follows if both $|g(x)|$ and $|g(y)|$ are larger than $|f(\dot{x})\wedge f(\dot{y})|$. This is the case because $f(x)\wedge f(y)\in g[T(\leq n)]$ and $g[T(n)]<l(n-1)<g[T(n+1)]$ according to Item ii. and because $g(x),g(y)\in g[T(n+1)]$.

If $\dot{x}=\dot{y}$ then it follows from $x\prec y$ that $|g(x)|>|g(y)|$ according to Item iv. The sequences $g(x)$ and $g(y)$ are predecessors of the sequence $f(\dot{x})$ with $f(x)\wedge f(\dot{x})=g(x)$ and $f(y)\wedge f(\dot{x})=g(y)$. Hence $f(x)\wedge f(y)=g(y)=f(\dot{y})\wedge f(y)=f(\dot{x})\wedge f(y)$. We have $f(x)(|f(x)\wedge f(y)|)=f(\dot{x})(|f(x)\wedge f(y)|)=f(\dot{y})(|f(x)\wedge f(y)|)=f(\dot{y})(|g(y)|)=0$ and $f(y)(|f(x)\wedge f(y)|)=f(y)(|g(y)|=1$. ∎

Because f restricted to $\{\langle\rangle\}$ is a strong diagonalization, Claim 3 implies that f restricted to $T(n)$ for every $n\in\omega$ is a strong diagonalization, which in turn implies that f is a strong diagonalization. ∎

5. On Milliken's result

Theorem 5.1 and Theorem 5.2 are due to K. Milliken, see [6]. We prove Theorem 5.2 as a consequence of Theorem 5.1 because we need it stated in our notation and the translation from the notation in Millikens paper is a bit cumbersome. Also, we think that the proof provided here is preferable.

The set T of sequences is *closed by levels* if for every $t\in T$ and $s\subseteq t$ with $|s|\in \mathrm{levels}(T)$ the sequence s is an element of T.

Let T be a meet closed set of sequences which is also closed by levels and let $n\in\omega+1$. The set $S\subseteq T$ is an element of $\mathrm{Str}^n(T)$ if:

- $|\mathrm{levels}(S)|=n$.
- S is meet closed and closed by levels.
- For all $s\in S$, the degree of s in S is equal to the degree of s in T, unless $\mathrm{levels}(S)$ has a maximum which is equal to $|s|$.

Note that if $S\in\mathrm{Str}^n(T)$ and $R\in\mathrm{Str}^m(S)$ then $R\in\mathrm{Str}^m(T)$.

Theorem 5.1 **(Milliken).** *Let $m,n\in\omega$ and T be an ω-tree. If*

$$\mathrm{Str}^n(T)=\bigcup_{i\in m} C_i$$

then there is $k\in m$ and

$$S\in\mathrm{Str}^\omega(T)$$

with

$$\mathrm{Str}^n(S)\subseteq C_k.$$

The elements of $\bigcup_{n\in\omega+1}\mathrm{Str}^n(T)$ are the *strong subsets* of T.

Let S be a meet closed finite subset of an ω-tree T with $n=|\mathrm{levels}(S)|$. A *cover* C of S in T is an element of $\mathrm{Str}^n(T)$ so that $S\subseteq C$ and $\mathrm{levels}(S)=\mathrm{levels}(C)$. Note that if C is a cover of S then $\nu_S[S]\subseteq\nu_C[C]$.

Lemma 5.1. *Let $S\overset{s}{\sim}R$ be two meet closed subsets of an ω-tree T. If there exists a subset C of T which is a cover of S and a cover of R then $S=R$.*

Proof. We have $\overrightarrow{\mathrm{levels}}(S)=\overrightarrow{\mathrm{levels}}(C)=\overrightarrow{\mathrm{levels}}(R):=l$ and $\nu_S[S]=\nu_R[R]\subseteq\nu_C[C]$ from Item 14. of Lemma 3.1. It suffices to show that if $s\in S$ and $r\in R$ with $s\circ l=r\circ l$ then $s=r$. This is the case according to Lemma 3.1 Item 9. because C is meet closed. ∎

Lemma 5.2. *Let T be an ω-tree and S a finite meet closed subset of T. Then there exists a cover C of S in T.*

Proof. Let R be a meet closed subset of T and $\overrightarrow{\text{levels}}(R)=\langle l_0,l_1,\dots,l_{n-1}\rangle$ for some $n\in\omega$ and let $r\in R$ with $|r|=l_k$ for some $k<n-1$. Let $N\subset\omega$ so that for every $i\in n$ there is $s\in R$ with $\langle r;i\rangle\subseteq s$. Let $j\in\omega\setminus N$ and $a\in T$ so that $|a|=l_{k+1}$ and $\langle r;j\rangle\subseteq a$. Then $R\cup\{a\}$ is meet closed and $\overrightarrow{\text{levels}}(R)=\overrightarrow{\text{levels}}(R\cup\{a\})$.

We say that $R\cup\{a\}$ *is constructed by degree completion.* The set R is *degree complete* if there is no $a\in T$ so that $R\cup a$ can be constructed by degree completion. Note that if R is degree complete then it has the property that for all $r\in R$, the degree of r in R is equal to the degree of r in T, unless $|r|=l_{n-1}$.

Let $r\in R$ and $|r|>i\in\omega$ so that the initial segment a of r with $|a|=l_i$ is not an element of R. Then $R\cup\{a\}$ is meet closed and $\overrightarrow{\text{levels}}(R)=\overrightarrow{\text{levels}}(R\cup\{a\})$. We say that $R\cup\{a\}$ *is constructed by level completion.* Note that if there is no $a\in T$ so that $R\cup a$ can be constructed by level completion then R is closed by levels.

It follows that if R is degree complete and closed by levels then R is a strong subset of T.

Because R has finitely many levels and the degree of every element of T is finite there is a subset C of T with $S\subseteq C$ and which is constructed from S via a sequence of successive level completions and degree completions and which is closed by levels and degree complete. Because $\overrightarrow{\text{levels}}(C)=\overrightarrow{\text{levels}}(S)$ it follows that $C\in\text{Str}^n(T)$. ∎

Theorem 5.2. *Let $F\in\mathfrak{T}_\omega$ be finite and meet closed, $m\in\omega$ and T be an ω-tree. If*

$$\text{Sims}_T(F)=\bigcup_{i\in m}C_i$$

then there is $k\in m$ and

$$S\in\text{Str}^\omega(T)$$

with

$$\text{Sims}_S(F)\subseteq C_k.$$

Proof. Let $n:=|\overrightarrow{\text{levels}}(F)|$. According to Lemma 5.2 there is a cover $f(G)$ for every $G\in\text{Sims}_T(F)$ of F and because of Lemma 5.1 the function f is an injection. Let $A=\text{Str}^n(T)\setminus f[\text{Sims}_T(F)]$ and $C_0^*:=f[C_0]\cup A$ and $C_i^*=f[C_i]$ for $1<i<m$. Then $\text{Str}^n(T)=\bigcup_{i\in m}C_i^*$.

According to Milliken's Theorem there is $k\in m$ and $S\in\text{Str}^\omega(T)$ so that $\text{Str}^n(S)\subseteq C_k^*$. Because f is an injection it follows that $\text{Sims}_S(F)\subseteq C_k$. ∎

6. The partition result for strong diagonalization's

Lemma 6.1. *Let F, G be two strongly diagonal sets. Then $F \overset{s}{\sim} G$ if and only if* $\mathrm{closure}(F) \overset{s}{\sim} \mathrm{closure}(G)$ *and $F = G$ if and only if* $\mathrm{closure}(F) = \mathrm{closure}(G)$.

Proof. Let $F \overset{s}{\sim} G$. It follows from Lemma 3.1 Item 3. that $\mathrm{closure}(F) \overset{s}{\sim} \mathrm{closure}(G)$. Let $\mathrm{closure}(F) \overset{s}{\sim} \mathrm{closure}(G)$. Then F is the set of endpoints of $\mathrm{closure}(F)$ and G the set of endpoints of $\mathrm{closure}(G)$. Checking Definition 3.1 we see that $F \overset{s}{\sim} G$.

If $F = G$ then clearly $\mathrm{closure}(F) = \mathrm{closure}(G)$. If $\mathrm{closure}(F) = \mathrm{closure}(G)$ then $F = G$ because F is the set of endpoints of $\mathrm{closure}(F) = \mathrm{closure}(G)$ and so is G. ∎

Theorem 6.1. *Let $F \in \mathfrak{T}_\omega$ be strongly diagonal, $m \in \omega$ and T be an ω-tree. If*

$$\mathrm{Sims}_T(F) = \bigcup_{i \in m} C_i$$

then there is $k \in m$ and

$$S \in \mathrm{Str}^\omega(T)$$

with

$$\mathrm{Sims}_S(F) \subseteq C_k.$$

Proof. Follows from Theorem 5.2 and Lemma 6.1. ∎

Lemma 6.2. *Let f be a strong diagonalization of the wide ω-tree T and G a subset of $f[T]$ so that $f^{-1}[G]$ is a strongly diagonal subset of T. Then $G \overset{s}{\sim} f^{-1}[G]$.*

Proof. It follows from Corollary 3.1 that $f^{-1}[G] \sim f[f^{-1}[G]] = G$ and from Lemma 3.6 that $G \overset{s}{\sim} f^{-1}[G]$. ∎

Corollary 6.1. *Let f be a strong diagonalization of the wide ω-tree T and let F, G be two finite subsets of $f[T]$ with $f^{-1}[F]$ and $f^{-1}[G]$ strongly diagonal. Then $f^{-1}[F] \overset{s}{\sim} f^{-1}[G]$ if and only if $F \overset{s}{\sim} G$.*

Lemma 6.3. *Let T be a wide ω-tree and $S \in \mathrm{Str}^\omega(T)$. Then there exists a strong similarity embedding of T into S.*

Proof. The following Claim establishes the Lemma.

Claim. *If f is a strong similarity embedding of $T(<n)$ for some $1 \leq n \in \omega$, that is if f is a strong similarity embedding of the union of the first n levels of T into S, then there is an extension f' of f to a strong similarity embedding of $T(\leq n)$ into S.*

Proof of Claim. Note that $|f(x)| \geq |x|$ for every $x \in T(<n)$ and hence because T is a wide ω-tree the degree of $f(x)$ in T is larger than or equal to the degree of x in T. Let $x \in T(n-1)$ with k_x equal to the degree of x. Then $\{\langle f(x); i\rangle \mid i \in k_x\}$ is a subset of the set of all immediate successors of $f(x)$.

For every $i \in k_x$ there is a unique element $y_{i,x} \in S$ so that $\langle f(x); i\rangle \subseteq y_{i,x}$ and if $z \in S$ with $\langle f(x); i\rangle \subseteq z$ then $y_{i,x} \subseteq z$. It follows from the definition of $\mathrm{Str}^\omega(T)$ and the fact that the strong similarity f maps all elements of L_{n-1} into the same level of S and hence into the same level of T, that $|y_{i,x}| = |y_{j,z}|$ for all $i \in k_x, j \in k_z$ and $x, z \in T(n-1)$.

Let f' be the extension of f to R' with $f(\langle x; i\rangle) = y_{i,x}$ for all $x \in T(n-1)$ and $i \in k_x$. Then f' is a strong similarity embedding of $T(\leq n)$ into S. ∎

Theorem 6.2. *Let T be a wide ω-tree, let f be a strong diagonalization of T, let A be a finite subset of $f[T]$ and $C_0 \cup C_1 \cup \cdots \cup C_{m-1} = \mathrm{Sims}_{f[T]}(A)$ be a partition of $\mathrm{Sims}_{f[T]}(A)$. Then there is $k \in m$ and a strong diagonalization g of T with $g[f[T]] \subseteq f[T]$ so that*

$$\mathrm{Sims}_{g \circ f[T]}(A) \subseteq C_k\,.$$

Proof. Let M be the set of elements F in $\mathrm{Sims}_{f[T]}(A)$ so that $f^{-1}[F]$ is strongly diagonal. For $i \in m-1$ let $D_i := \{f^{-1}[C] \mid C \in C_i \cap M\}$ and put $D_{m-1} := \{f^{-1}[C] \mid C \in C_{m-1} \cap M)\} \cup (\mathrm{Sims}_T(A) \setminus \{f^{-1}[C] \mid C \in M\}$. Then $D_0, D_1, \ldots, D_{m-1}$ is a partition of $\mathrm{Sims}_T(A)$.

Using Theorem 6.1 there is $S \in \mathrm{Str}^\omega(T)$ and $k \in m$ with $\mathrm{Sims}_S(A) \subseteq D_k$. According to Lemma 6.3 there exists a strong similarity embedding h of T into S. Let $V = f[S]$. It follows from Lemma 6.2 and the definition of V that if $F \in M$ and $F \subseteq V$ then $F \in C_k$.

Let $g := f \circ h$ then g is a strong diagonalization of T according to Lemma 3.8 and maps $f[T]$ into $f[T]$ and $g \circ f[T] = f \circ h \circ f[T] := W \subseteq V$. The function $h \circ f$ is a strong diagonalization of T according to Lemma 3.8. Hence if $F \in \mathrm{Sims}_W(A)$ then $f^{-1}[F] \in h \circ f[T]$ is strongly diagonal and it follows that $F \in M$ which in turn implies that $F \in C_k$. ∎

Theorem 6.3. *Let D be a cofinal subset of the wide ω-tree T and A a finite strongly diagonal subset of D. Let $C_0 \cup C_1 \cup \cdots \cup C_{m-1} = \mathrm{Sims}_D(A)$ be a partition of $\mathrm{Sims}_D(A)$. Then there is $k \in m$ and a strong diagonalization h of T into D so that*

$$\mathrm{Sims}_{h[D]}(A) \subseteq C_k\,.$$

Proof. Let f be a strong diagonalization of T into D and for $i \in m$ let $C'_i = \{F \in C_i \mid F \subseteq f[T]\}$. It follows that $C'_0 \cup C'_1 \cup \cdots \cup C'_{m-1} = \mathrm{Sims}_{f[T]}(A)$ is a partition of $\mathrm{Sims}_{f[T]}(A)$. According to Theorem 6.2 there exists a number

$k \in m$ and a strong diagonalization g of T with $g[f[T]] \subseteq f[T] \subseteq D$ so that $\mathrm{Sims}_{g\circ f[T]}(A) \subseteq C_k$. Let $h = g \circ f$. ∎

Theorem 6.4. *Let D be a cofinal subset of the wide ω-tree T and A a finite diagonal subset of D. Let $C_0 \cup C_1 \cup \cdots \cup C_{m-1} = \mathrm{Sim}_D(A)$ be a partition of $\mathrm{Sim}_D(A)$. Then there is $k \in m$ and a strong diagonalization h of T so that*

$$\mathrm{Sim}_{h[D]}(A) \subseteq C_k\,.$$

Proof. Let f be a strong diagonalization of T into D. It follows from Corollary 3.1 that $A' = f[A]$ is strongly diagonal with $A' \sim A$. Then $\mathrm{Sim}_D(A) = \mathrm{Sim}_D(A')$.

For $i \in m$ let $C'_i = \{F \in C_i \mid F \subseteq f[T]\}$. It follows that $C'_0 \cup C'_1 \cup \cdots \cup C'_{m-1} = \mathrm{Sim}_{f[T]}(A')$ is a partition of $\mathrm{Sim}_{f[T]}(A')$. It follows from the fact that $f[T]$ is strongly diagonal and from Lemma 3.6 that $\mathrm{Sims}_{f[T]}(A') = \mathrm{Sim}_{f[T]}(A')$. According to Theorem 6.2 there exists a number $k \in m$ and a strong diagonalization g of T with $g[f[T]] \subseteq f[T] \subseteq D$ so that $\mathrm{Sims}_{g\circ f[T]}(A') \subseteq C_k$. Let $h = g \circ f$. ∎

7. Proof of Theorem 2.1

Theorem 2.1. *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable binary relational structure and F a finite diagonal subset of U. Let $C_0, C_1, \ldots, C_{m-1}$ be a partition of $\mathrm{Sim}_U(F)$ into equivalence classes.*

Then there exists $k \in m$ and a diagonalization f of $\mathbb{U}$ into $\mathbb{U}$ so that $\mathrm{Sim}_{f[U]}(F) \subseteq C_k$.

Proof. Let $\mathbf{F}$ be a universal constraint set and $\mathbb{U} = \mathbb{U}_{\mathbf{F}}$ the universal homogeneous structures satisfying the constraints $\mathbf{F}$. Let $|\mathbf{F}| = k \in \omega$ and λ a bijection of $\mathbf{F}$ to k and let $(u_i; i \in \omega)$ be an enumeration of the elements of U. Let T be the regular ω-tree of degree k and $\mathbb{F}$ the relational structure with base set T so that the function σ described in Section 2 is an isomorphic embedding of $\mathbb{U}_{\mathbf{F}}$ into $\mathbb{T}_{\mathbf{F}}$.

Then $\sigma[U]$ is a cofinal subset of T. We see this as follows:

Let $s = \langle s_0, s_1, \ldots, s_{n-1}\rangle \in T$. Let x be an element not in U and $\mathbb{A} = (\{u_i \mid i \in n\} \cup \{x\}; \mathfrak{L})$ be a relational structure in language $\mathfrak{L}$ and base set $\{u_i \mid i \in n\} \cup \{x\}$ so that $\mathbb{A}$ restricted to $\{u_i \mid i \in n\}$ is equal to $\mathbb{U}$ restricted to $\{u_i \mid i \in n\}$ and so that $\lambda(\mathbb{F}) = s_i$ where $\mathbb{F} \in \mathbf{F}$ is isomorphic the restriction of $\mathbb{A}$ to $\{u_i, x\}$. Then $\mathbb{A}$ is an element of the age of $\mathbb{U}_{\mathbf{F}}$.

We obtain, from the mapping extension property of $\mathbb{U}_{\mathbf{F}}$, an embedding f of $\mathbb{A}$ into $\mathbb{U}_{\mathbf{F}}$ which is the identity on the set $\{u_i \mid i \in n\}$. Let $f(x) = u_l$.

Note that $l \geq n$ because f is an injection. It follows that s is a predecessor of $\sigma(f(x)) = \sigma(u_l) \in \sigma[U]$ and hence that $\sigma[U]$ is cofinal in T.

Theorem 2.1 follows now from Theorem 6.4. ∎

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